

Strong Convergence of Euler Approximations of Stochastic Differential Equations with Delay under Local Lipschitz Condition

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March 7, 2013

Abstract

The strong convergence of Euler approximations of stochastic delay differential equations is proved under general conditions. The assumptions on drift and diffusion coefficients have been relaxed to include polynomial growth and only continuity in the arguments corresponding to delays. Furthermore, the rate of convergence is obtained under one-sided and polynomial Lipschitz conditions. Finally, our findings are demonstrated with the help of numerical simulations.

Keywords: stochastic delay differential equations, Euler approximations, strong convergence, rate of convergence, local Lipschitz condition.

1 Introduction

In modeling many real world phenomena, the future states of the system depend not only on the present state but also on its past state(s). The models based on stochastic delay differential equations (SDDEs) could be used in such situations. They have found numerous applications in various fields, for example, in communications, physics, biology, ecology, economics and finance. One could refer to [1] - [10] and references therein.

*The authors would like to thank Lukas Szpruch for his useful discussions.

In many applications, one finds SDDEs for which solutions can not be obtained explicitly and hence this necessitates the development of numerical schemes. Over the past few years, many authors have shown their interest in studying strong convergence properties of numerical schemes of SDDEs, to name a few, [11] - [16]. Recently there has been a growing interest in weak convergence of numerical schemes as well, see for example, [17], [18] and [19]. We also mention a paper by [20] on pathwise approximations and its relation to strong convergence. For the purpose of this article, we study strong convergence of Euler approximations of SDDEs due to its wide applicability. We exploit the fact that SDDEs can be regarded as special cases of stochastic differential equations (SDEs) with random coefficients. To this end, first we establish strong convergence results of Euler schemes of SDEs with random coefficients under mild conditions. This, in turn, allows us to prove the convergence in the case of SDDEs under more relaxed conditions than those existing in the literature. More precisely, to the best of our knowledge, strong convergence has been proved so far by assuming linear growth and local Lipschitz conditions in both variables corresponding to delay and non-delay arguments, see for example [16]. Whereas, we relax these conditions by assuming polynomial growth and continuity in the argument corresponding to delays. Moreover no smoothness condition on the initial data is assumed. Therefore, the setting in this article is more general than those present in the literature, see for example, [11], [14], [15], [16] and [20]. In addition, it is shown here that the rate of convergence is one-fourth when the drift coefficient satisfies one-sided Lipschitz and polynomial Lipschitz in the non-delay and delay arguments respectively. This result is in agreement with the findings of [21].

We conclude this section with the introduction of some basic notation. The norm of a vector $x \in \mathbb{R}^d$ and the Hilbert-Schmidt norm of a matrix $A \in \mathbb{R}^{d \times m}$ are respectively denoted by $|x|$ and $|A|$. The transpose of a vector $x \in \mathbb{R}^d$ is denoted by x^T and the inner product of two vectors $x, y \in \mathbb{R}^d$ is denoted by $\langle x, y \rangle := x^T y$. The integer part of a real number x is denoted by $[x]$. We denote the indicator function of set A by $\mathbb{1}_A$. Moreover, $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of random variables X with a norm $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p} < \infty$ for $p > 0$. Finally, \mathcal{P} denotes the predictable σ -algebra on $\mathbb{R}_+ \times \Omega$ and $\mathcal{B}(V)$, the σ -algebra of Borel sets of topological spaces V .

2 Main Result

Let $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e. the filtration is increasing and right continuous. Let $\{W(t)\}_{t \geq 0}$ be an m -dimensional Wiener martingale. Furthermore, it is assumed that $\beta(t, y_1, \dots, y_k, x)$ and $\alpha(t, y_1, \dots, y_k, x)$ are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^{d \times k}) \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions and take values in \mathbb{R}^d and $\mathbb{R}^{d \times m}$ respectively. For a fixed $T > 0$, let the stochastic delay differential equation (SDDE) on $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, \mathcal{F}, \mathbb{P})$ be defined as follows,

$$\begin{aligned} dX(t) &= \beta(t, Y(t), X(t))dt + \alpha(t, Y(t), X(t))dW(t), \quad t \in [0, T], \\ X(t) &= \xi(t), \quad t \in [-H, 0], \end{aligned} \tag{1}$$

where $\{\xi(t) : -H \leq t \leq 0\} \in \mathcal{C}_{\mathcal{F}_0}^b([-H, 0]; \mathbb{R}^d)$ for some $H > 0$ and $Y(t) := (X(\delta_1(t)), \dots, X(\delta_k(t)))$. The delay parameters $\delta_1(t), \dots, \delta_k(t)$ are increasing functions of t and satisfy $-H \leq \delta_j(t) \leq [\frac{t}{\tau}] \tau$ for some $\tau > 0$ and $j = 1, \dots, k$.

Further, for every $n \geq 1$ and $t \in [0, T]$, the Euler scheme of SDDE (1) is given by

$$\begin{aligned} dX_n(t) &= \beta(t, Y_n(t), X_n(\kappa_n(t)))dt + \alpha(t, Y_n(t), X_n(\kappa_n(t)))dW(t), \quad t \in [0, T], \\ X_n(t) &= \xi(t), \quad t \in [-H, 0], \end{aligned} \tag{2}$$

where $Y_n(t) := (X_n(\delta_1(t)), \dots, X_n(\delta_k(t)))$ and κ_n is defined by

$$\kappa_n(t) := \frac{[n(t - t_0)]}{n} + t_0 \tag{3}$$

with the observation that for equation (2) one takes $t_0 = 0$ in (3). We note that two popular cases of delay viz. $\delta_i(t) = t - \tau$ and $\delta_i(t) = [\frac{t}{\tau}] \tau$ can be addressed by our findings. This type of delay parameters have been extensively discussed and found wide applications in the literature. The reader could consult the following, for example, [1] - [6], [10] and references therein.

Remark 1. *We note that the Euler scheme (2) defines approximations to SDDEs in an explicit way without a discretization of the delay terms. From our main theorem and corollaries on convergence of scheme (2), one could easily obtain results on convergence of this scheme with discretized delay terms. Therefore for matters of notational simplicity, we choose the former approach.*

Let $y := (y_1, \dots, y_k)$. We make the following assumptions for our result.

C-1. There exist constants $G > 0$ and $l > 0$ such that for any $t \in [0, T]$,

$$|\beta(t, y, x)| + |\alpha(t, y, x)| \leq G(1 + |y|^l + |x|)$$

for all $x \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d \times k}$.

C-2. For every $R > 0$, there exists a constant $K_R > 0$ such that for any $t \in [0, T]$,

$$\langle x - z, \beta(t, y, x) - \beta(t, y, z) \rangle \vee |\alpha(t, y, x) - \alpha(t, y, z)|^2 \leq K_R |x - z|^2$$

whenever $|x|, |y|, |z| < R$.

C-3. The functions $\beta(t, y, x)$ and $\alpha(t, y, x)$ are continuous in y uniformly in x from compacts, i.e. for every $R > 0$ and $t \in [0, T]$,

$$\sup_{|x| \leq R} \{|\beta(t, y, x) - \beta(t, y', x)| + |\alpha(t, y, x) - \alpha(t, y', x)|\} \rightarrow 0 \quad \text{as } y' \rightarrow y.$$

The conditions C-1 and C-2 are sufficient for existence and uniqueness of solution of SDDE (1) and the Euler scheme (2)(see [12], [22]). We state the main result of this paper in the following theorem.

Theorem 1. *Suppose C-1 to C-3 hold, then the Euler scheme (2) converges to the true solution of SDDE (1) in \mathcal{L}^p -sense, i.e.*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] = 0$$

for all $p > 0$.

Remark 2. *We note that we can assume, without loss of generality, that T is a multiple of τ . If not, then SDDE (1) and its EM scheme (2) are defined for $T' > T$ with $T' = N\tau$, where N is a positive integer. The results given in this report are then recovered for the original SDDE (1) by choosing drift and diffusion parameters as $\beta 1_{\{t \leq T\}}$ and $\alpha 1_{\{t \leq T\}}$.*

First, we develop requisite theory for SDEs with random coefficients in the following section and then prove Theorem 1 in Section 4.

3 SDEs with Random Coefficients

Let $b(t, x)$ and $\sigma(t, x)$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions which take values in \mathbb{R}^d and $\mathbb{R}^{d \times m}$ respectively. Suppose $0 \leq t_0 < t_1 \leq T$, then let us consider an SDE with random coefficients given by

$$dX(t) = b(t, X(t))dt + \sigma(t, X(t))dW(t), \quad \forall t \in [t_0, t_1], \quad (4)$$

with initial value $X(t_0)$ which is an almost surely finite \mathcal{F}_{t_0} -measurable random variable.

For every $n \geq 1$, let $b_n(t, x)$ and $\sigma_n(t, x)$ be $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable functions which take values in \mathbb{R}^d and $\mathbb{R}^{d \times m}$ respectively. Then for any $t \in [t_0, t_1]$, we define the following Euler scheme corresponding to the SDE (4) as,

$$dX_n(t) = b_n(t, X_n(\kappa_n(t)))dt + \sigma_n(t, X_n(\kappa_n(t)))dW(t) \quad (5)$$

with initial value $X_n(t_0)$, which is an \mathcal{F}_{t_0} -measurable, almost surely finite, random variable and κ_n as defined in (3).

We make the following assumptions.

A-1. There exists a constant $c > 0$ and non-negative random variables $\{M_n\}_{n \geq 1}$ and M with $\mathbb{E}[M^p] \vee \sup_{n \geq 1} \mathbb{E}[M_n^p] < N$ for every $p > 0$ and some $N := N(p) > 0$ such that, almost surely,

$$|b(t, x)| + |\sigma(t, x)| \leq c(M + |x|)$$

$$|b_n(t, x)| + |\sigma_n(t, x)| \leq c(M_n + |x|)$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$.

A-2. For every $R > 0$, there exists an \mathcal{F}_{t_0} -measurable random variable C_R such that, almost surely, for any $t \in [t_0, t_1]$,

$$\langle x - y, b(t, x) - b(t, y) \rangle \vee |\sigma(t, x) - \sigma(t, y)|^2 \leq C_R |x - y|^2$$

for all $|x|, |y| \leq R$. Moreover, there exists a function $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$\lim_{R \rightarrow \infty} \mathbb{P}(C_R > f(R)) = 0.$$

A-3. For every $R > 0$ and $p > 0$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{t_0}^{t_1} \sup_{|x| \leq R} \{ |b_n(t, x) - b(t, x)|^p + |\sigma_n(t, x) - \sigma(t, x)|^p \} dt \right] = 0.$$

A-4. The initial values of SDE (4) and its Euler scheme (5) satisfy

$$\lim_{n \rightarrow \infty} \mathbb{E}[|X(t_0) - X_n(t_0)|^p] = 0$$

for every $p > 0$.

A-5. The initial values of SDE (4) and its Euler scheme (5) have bounded p -th moments, i.e. there exists $L := L(p) > 0$ such that

$$\mathbb{E}[|X(t_0)|^p] \vee \sup_{n \geq 1} \mathbb{E}[|X_n(t_0)|^p] < L$$

for every $p > 0$.

We note that assumptions A-1 and A-2 are sufficient for existence and uniqueness of solution of SDE (4) (see [23] and [24]). Before proving the main result (Theorem 2) of this section, we establish some lemmas.

Lemma 1. *Suppose that A-1 and A-5 hold, then for some $K := K(p, T, L) > 0$,*

$$\begin{aligned} & \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t)|^p \right] < K \\ \text{and} \quad & \sup_{n \geq 1} \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X_n(t)|^p \right] < K \end{aligned} \tag{6}$$

for every $p > 0$.

Proof. First one chooses any $p > 2$. Then for the true solution $\{X(t)\}_{t \in [t_0, t_1]}$, by using Hölder's

inequality, one obtains,

$$|X(t)|^p \leq 3^{p-1} \left\{ |X(t_0)|^p + |t - t_0|^{p-1} \int_{t_0}^t |b(s, X(s))|^p ds + \left| \int_{t_0}^t \sigma(s, X(s)) dW(s) \right|^p \right\}.$$

Thus on taking supremum over $[t_0, u]$ for some $u \in [t_0, t_1]$ and by applying Burkholder-Davis-Gundy inequality with constant $\bar{c} := \bar{c}(p) > 0$, one observes that

$$\mathbb{E} \left[\sup_{t_0 \leq t \leq u} |X(t)|^p \right] \leq 3^{p-1} \left\{ \mathbb{E}[|X(t_0)|^p] + T^{p-1} \mathbb{E} \left[\int_{t_0}^u |b(s, X(s))|^p ds \right] + \bar{c} \mathbb{E} \left[\int_{t_0}^u |\sigma(s, X(s))|^p ds \right] \right\}.$$

Now A-1 and A-5 give

$$\begin{aligned} \mathbb{E} \left[\sup_{t_0 \leq t \leq u} |X(t)|^p \right] &\leq 2^{p-1} \left\{ \mathbb{E}[|X(t_0)|^p] + [T^{p-1} + \bar{c}] c^p \mathbb{E} \left[\int_{t_0}^u \{|M| + |X(s)|\}^p ds \right] \right\} \\ &\leq 3^{p-1} \left\{ \mathbb{E}[|X(t_0)|^p] + 2^{p-1} [T^{p-1} + \bar{c}] c^p T N \right. \\ &\quad \left. + 2^{p-1} [T^{p-1} + \bar{c}] c^p \int_{t_0}^u \mathbb{E} \left[\sup_{t_0 \leq r \leq s} |X(r)|^p \right] ds \right\} \end{aligned}$$

which on the application of Gronwall's inequality yields

$$\mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t)|^p \right] < 3^{p-1} \left\{ \mathbb{E}[|X(t_0)|^p] + 2^{p-1} [T^{p-1} + \bar{c}] c^p T N \right\} e^{6^{p-1} (T^{p-1} + \bar{c}) c^p T}.$$

For any $p > 2$, we adopt similar arguments for $\{X_n(t)\}_{t \in [t_0, t_1]}$ in order to prove (6) with the remark that A-5 guarantees that K does not depend on n . Then one observes that the application of Hölder's inequality completes the proof. \square

For every $R > 0$ and $n \geq 1$, let us define the stopping times,

$$\tau_R := \inf\{t \geq t_0 : |X(t)| \geq R\}, \quad \sigma_{nR} := \inf\{t \geq t_0 : |X_n(t)| \geq R\} \quad \text{and} \quad \nu_{nR} := \tau_R \wedge \sigma_{nR}, \quad (7)$$

where $\inf \emptyset = \infty$.

We prove below a very useful lemma of this article as it is used in the penultimate section in order to recover the rate of convergence.

Lemma 2. *Let us assume that A-1 and A-5 hold. Then*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] = 0$$

for every $p > 0$.

Proof. In order to prove the result, one chooses $p \geq 2$. Then one immediately writes,

$$\begin{aligned} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] &= \mathbb{E} \left[\int_{t_0}^{t_1} \left| \int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} b_n(r, X_n(\kappa_n(r))) dr \right. \right. \\ &\quad \left. \left. + \int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} \sigma_n(r, X_n(\kappa_n(r))) dW(r) \right|^p ds \right] \end{aligned}$$

which implies on applying Hölder's inequality

$$\begin{aligned} &\mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] \\ &\leq 2^{p-1} \mathbb{E} \left[\int_{t_0}^{t_1} |s \wedge \nu_{nR} - \kappa_n(s \wedge \nu_{nR})|^{p-1} \left(\int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} |b_n(r, X_n(\kappa_n(r)))|^p dr \right) ds \right] \\ &\quad + 2^{p-1} \int_{t_0}^{t_1} \mathbb{E} \left[\left| \int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} \sigma_n(r, X_n(\kappa_n(r))) dW(r) \right|^p \right] ds \\ &\leq \left(\frac{2}{n} \right)^{p-1} \int_{t_0}^{t_1} \mathbb{E} \left[\int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} |b_n(r, X_n(\kappa_n(r)))|^p dr \right] ds \\ &\quad + 2^{p-1} \int_{t_0}^{t_1} \mathbb{E} \left[\left| \int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} \sigma_n(r, X_n(\kappa_n(r))) dW(r) \right|^p \right] ds. \end{aligned} \tag{8}$$

One finds bounds for the integrand of the first term of (8) as follows,

$$\begin{aligned} &\mathbb{E} \left[\int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} |b_n(r, X_n(\kappa_n(r)))|^p dr \right] \leq c^p \mathbb{E} \left[\int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} (|M_n| + |X_n(\kappa_n(r))|)^p dr \right] \\ &\leq c^p 2^{p-1} \mathbb{E} \left[\int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} (|M_n|^p + |X_n(\kappa_n(r))|^p) dr \right] \\ &\leq c^p 2^{p-1} \mathbb{E} \left[\left(|M_n|^p + \sup_{t_0 \leq r \leq t_1} |X_n(r)|^p \right) |s \wedge \nu_{nR} - \kappa_n(s \wedge \nu_{nR})| \right] \\ &\leq \frac{c^p 2^{p-1}}{n} \left(\mathbb{E}[|M_n|^p] + \mathbb{E} \left[\sup_{t_0 \leq r \leq t_1} |X_n(r)|^p \right] \right) \\ &\leq \frac{c^p 2^{p-1}}{n} (K + N). \quad (\text{using Lemma 1}) \end{aligned} \tag{9}$$

For the second term of (8), one writes,

$$\begin{aligned} \mathbb{E} \left[\left| \int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} \sigma_n(r, X_n(\kappa_n(r))) dW(r) \right|^p \right] &= \mathbb{E} \left[\left| \int_{\kappa_n(s)}^s \mathbb{1}_{[\kappa_n(s \wedge \nu_{nR}), s \wedge \nu_{nR}]} \sigma_n(r, X_n(\kappa_n(r))) dW(r) \right|^p \right] \\ &\leq \left(\frac{p(p-1)}{2} \right)^{p/2} |s - \kappa_n(s)|^{(p-2)/2} \mathbb{E} \left[\int_{\kappa_n(s)}^s \mathbb{1}_{[\kappa_n(s \wedge \nu_{nR}), s \wedge \nu_{nR}]} |\sigma_n(r, X_n(\kappa_n(r)))|^p dr \right] \end{aligned}$$

which on the application of A-1 and Lemma 1 yields

$$\mathbb{E} \left[\left| \int_{\kappa_n(s \wedge \nu_{nR})}^{s \wedge \nu_{nR}} \sigma_n(r, X_n(\kappa_n(r))) dW(r) \right|^p \right] \leq \left(\frac{p(p-1)}{2} \right)^{p/2} \frac{1}{n^{p/2}} c^p 2^{p-1} (K + N). \quad (10)$$

Substituting (9) and (10) in (8) yields

$$\begin{aligned} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] &\leq \frac{T(K + N)4^{p-1}c^p}{n^{p/2}} \left\{ \frac{1}{n^{p/2}} + \left(\frac{p(p-1)}{2} \right)^{p/2} \right\} \\ &\leq \mathcal{O}(n^{-p/2}). \end{aligned} \quad (11)$$

Then by letting $n \rightarrow \infty$ on both sides, one obtains

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] = 0$$

for all $p \geq 2$. Then one observes that the application of Hölder's and Jensen's inequalities completes the proof. \square

In order to prove the following theorem, a similar approach to the one adopted by [25] for SDEs is followed here, but in a more general context.

Theorem 2. *Let A-1 to A-5 hold and suppose there exists a unique solution $\{X_n(t)\}_{\{t_0 \leq t \leq t_1\}}$ of the Euler scheme (5). Then, the Euler scheme (5) converges to the SDE (4) in \mathcal{L}^p -sense, i.e.*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t) - X_n(t)|^p \right] = 0 \quad (12)$$

for all $p > 0$.

Proof. For every $R > 0$ and $n \geq 1$, we consider stopping times as defined in (7). First we fix $p > 4$.

Then

$$\begin{aligned} \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t) - X_n(t)|^p \right] &\leq \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t) - X_n(t)|^p \mathbb{1}_{\{\tau_R \leq t_1 \text{ or } \sigma_{nR} \leq t_1 \text{ or } C_R > f(R)\}} \right] \\ &\quad + \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t \wedge \nu_{nR}) - X_n(t \wedge \nu_{nR})|^p \mathbb{1}_{\{C_R \leq f(R)\}} \right]. \end{aligned} \quad (13)$$

To estimate the first term of (13), one can use Young's inequality for $q > p$ ($1/p + 1/q = 1$) and $\eta > 0$ to obtain the following.

$$\begin{aligned} \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t) - X_n(t)|^p \mathbb{1}_{\{\tau_R \leq t_1 \text{ or } \sigma_{nR} \leq t_1 \text{ or } C_R > f(R)\}} \right] &\leq \frac{\eta p}{q} \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t) - X_n(t)|^q \right] \\ &\quad + \frac{q-p}{q\eta^{p/(q-p)}} \mathbb{P}(\tau_R \leq t_1 \text{ or } \sigma_{nR} \leq t_1 \text{ or } C_R > f(R)) \\ &\leq \frac{\eta p}{q} 2^{q-1} \left\{ \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t)|^q \right] + \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X_n(t)|^q \right] \right\} \\ &\quad + \frac{q-p}{q\eta^{p/(q-p)}} \left\{ \mathbb{P}(\tau_R \leq t_1) + \mathbb{P}(\sigma_{nR} \leq t_1) + \mathbb{P}(C_R > f(R)) \right\} \\ &\leq \frac{\eta p}{q} 2^q K + \frac{q-p}{q\eta^{p/(q-p)}} \left\{ \mathbb{E} \left[\frac{|X(\tau_R)|^p}{R^p} \right] + \mathbb{E} \left[\frac{|X_n(\sigma_{nR})|^p}{R^p} \right] + \mathbb{P}(C_R > f(R)) \right\}, \end{aligned}$$

which becomes

$$\begin{aligned} \mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |X(t) - X_n(t)|^p \mathbb{1}_{\{\tau_R \leq t_1 \text{ or } \sigma_{nR} \leq t_1 \text{ or } C_R > f(R)\}} \right] \\ \leq \frac{\eta p}{q} 2^q K + \frac{q-p}{q\eta^{p/(q-p)}} \left\{ \frac{2K}{R^p} + \mathbb{P}(C_R > f(R)) \right\}. \end{aligned} \quad (14)$$

Further for estimating the second term of (13), let $e_n(s) := X(s) - X_n(s)$ for $s \in [t_0, t_1]$, then one could write

$$de_n(s) = \bar{b}_n(s)ds + \bar{\sigma}_n(s)dW(s)$$

where $\bar{b}_n(s) := b(s, X(s)) - b_n(s, X_n(\kappa_n(s)))$ and $\bar{\sigma}_n(s) := \sigma(s, X(s)) - \sigma_n(s, X_n(\kappa_n(s)))$. Thus, on the application of Itô's formula, one obtains

$$d|e_n(s)|^2 = 2\langle e_n(s), \bar{b}_n(s) \rangle ds + \sum_{i=1}^m |\bar{\sigma}_n^i(s)|^2 ds + 2 \sum_{i=1}^m \langle e_n(s), \bar{\sigma}_n^i(s) \rangle dW^i(s)$$

which on integrating over $s \in [t_0, t \wedge \nu_{nR}]$ for some $t \in [t_0, t_1]$ yields

$$\begin{aligned} |e_n(t \wedge \nu_{nR})|^2 &= |e_n(t_0)|^2 + 2 \int_{t_0}^{t \wedge \nu_{nR}} \langle e_n(s), \bar{b}_n(s) \rangle ds + \sum_{i=1}^m \int_{t_0}^{t \wedge \nu_{nR}} |\bar{\sigma}_n^i(s)|^2 ds \\ &\quad + 2 \sum_{i=1}^m \int_{t_0}^{t \wedge \nu_{nR}} \langle e_n(s), \bar{\sigma}_n^i(s) \rangle dW^i(s) \end{aligned}$$

where σ_n^i and W^i denote the i -th column of $(d \times m)$ -matrix σ and the i -th element of Wiener $(m \times 1)$ -vector W respectively for all $i \in \{1, \dots, m\}$. Further, after raising the power to $p/2$ for some $p > 4$ and applying Hölder's and Burkholder-Davis-Gundy inequalities, one obtains the following,

$$\begin{aligned} \mathbb{E} \left[\sup_{t_0 \leq t \leq u} |e_n(t \wedge \nu_{nR})|^p \mathbb{1}_{\{C_R \leq f(R)\}} \right] &\leq 2^{p-2} \mathbb{E}[|e_n(t_0)|^p] \\ &\quad + 2(8T)^{\frac{p-2}{2}} \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_{nR}} |\langle e_n(s), \bar{b}_n(s) \rangle|^{\frac{p}{2}} ds \right] \\ &\quad + (4mT)^{\frac{p-2}{2}} \sum_{i=1}^m \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_{nR}} |\bar{\sigma}_n^i(s)|^p ds \right] \\ &\quad + 2^{\frac{3p-4}{2}} m^{\frac{p-2}{2}} T^{\frac{p-4}{4}} \bar{c} \sum_{i=1}^m \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_{nR}} |\langle e_n(s), \bar{\sigma}_n^i(s) \rangle|^{\frac{p}{2}} ds \right] \\ &= I + II + III + IV \end{aligned} \tag{15}$$

where $\bar{c} := \bar{c}(p)$ is the constant of the Burkholder-Davis-Gundy inequality.

In order to estimate II , one observes that the application of A-2 and Cauchy-Schwarz inequality yields,

$$\begin{aligned} \langle e_n(s), \bar{b}_n(s) \rangle &= \langle X(s) - X_n(\kappa_n(s)), b(s, X(s)) - b(s, X_n(\kappa_n(s))) \rangle \\ &\quad + \langle X(s) - X_n(\kappa_n(s)), b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s))) \rangle \\ &\quad + \langle X_n(\kappa_n(s)) - X_n(s), b(s, X(s)) - b_n(s, X_n(\kappa_n(s))) \rangle \\ &\leq C_R |X(s) - X_n(\kappa_n(s))|^2 + |X(s) - X_n(\kappa_n(s))| |b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s)))| \\ &\quad + |X_n(s) - X_n(\kappa_n(s))| |b(s, X(s)) - b_n(s, X_n(\kappa_n(s)))| \end{aligned} \tag{16}$$

and on further application of Young's inequality and A-1, this becomes

$$\begin{aligned} \langle e_n(s), \bar{b}_n(s) \rangle &\leq (2C_R + 1)|e_n(s)|^2 + (2C_R + 1)|X_n(s) - X_n(\kappa_n(s))|^2 \\ &\quad + \frac{1}{2}|b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s)))|^2 \\ &\quad + |X_n(s) - X_n(\kappa_n(s))| \{c(|X(s)| + |X(\kappa_n(s))| + M + M_n)\}. \end{aligned}$$

Therefore raising power $p/2$ on both sides, we get

$$\begin{aligned} |\langle e_n(s), \bar{b}_n(s) \rangle|^{\frac{p}{2}} &\leq 2^{p-3} \left\{ 2(2C_R + 1)^{\frac{p}{2}} |e_n(s)|^p + 2(2C_R + 1)^{\frac{p}{2}} |X_n(s) - X_n(\kappa_n(s))|^p \right. \\ &\quad \left. + |b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s)))|^p \right. \\ &\quad \left. + 2|X_n(s) - X_n(\kappa_n(s))|^{\frac{p}{2}} \{c(|X(s)| + |X(\kappa_n(s))| + M + M_n)\}^{\frac{p}{2}} \right\} \end{aligned}$$

for every $s \in [t_0, u \wedge \nu_{nR}]$. Hence one obtains the following estimate on using Hölder's inequality,

$$\begin{aligned} II &:= (32T)^{\frac{p-2}{2}} \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_{nR}} |\langle e_n(s), \bar{b}_n(s) \rangle|^{\frac{p}{2}} ds \right] \\ &\leq (32T)^{\frac{p-2}{2}} \left\{ 2(2f(R) + 1)^{\frac{p}{2}} \int_{t_0}^u \mathbb{E} \left[\sup_{t_0 \leq r \leq s} |e_n(r \wedge \nu_{nR})|^p \mathbb{1}_{C_R \leq f(R)} \right] ds \right. \\ &\quad \left. + 2(2f(R) + 1)^{\frac{p}{2}} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] \right. \\ &\quad \left. + \mathbb{E} \left[\int_{t_0}^{t_1} |b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - b_n(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \right. \\ &\quad \left. + 2^{p+1/2} T C^{\frac{p}{2}} \sqrt{\mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right]} \times \sqrt{K + N} \right\}. \end{aligned} \quad (17)$$

Now in order to estimate *III* of (15), one observes that for every $i \in \{1, \dots, m\}$, the assumption

A-2 yields

$$\begin{aligned} |\bar{\sigma}_n^i(s)|^p &= |\sigma^i(s, X(s)) - \sigma_n^i(s, X_n(\kappa_n(s)))|^p \leq 3^{p-1} \left\{ C_R^{\frac{p}{2}} |e_n(s)|^p + C_R^{\frac{p}{2}} |X_n(s) - X_n(\kappa_n(s))|^p \right. \\ &\quad \left. + |\sigma^i(s, X_n(\kappa_n(s))) - \sigma_n^i(s, X_n(\kappa_n(s)))|^p \right\} \end{aligned}$$

which implies

$$\begin{aligned}
III &:= (4mT)^{\frac{p-2}{2}} \sum_{i=1}^m \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_{nR}} |\bar{\sigma}_n^i(s)|^p ds \right] \\
&\leq 3^{p-1} (4mT)^{\frac{p-2}{2}} \left\{ mf(R)^{\frac{p}{2}} \int_{t_0}^u \mathbb{E} \left[\sup_{t_0 \leq r \leq s} |e_n(r \wedge \nu_{nR})|^p \mathbb{1}_{\{C_R \leq f(R)\}} \right] ds \right. \\
&\quad \left. + mf(R)^{\frac{p}{2}} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] \right. \\
&\quad \left. + \sum_{i=1}^m \mathbb{E} \left[\int_{t_0}^{t_1} |\sigma^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - \sigma_n^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \right\}. \quad (18)
\end{aligned}$$

Finally, to estimate IV , observes that for $i \in \{1, \dots, m\}$, an application of assumption A-2 yields

$$|\langle e_n(s), \bar{\sigma}_n^i(s) \rangle|^{\frac{p}{2}} \leq |e_n(s)|^{\frac{p}{2}} |\bar{\sigma}_n^i(s)|^{\frac{p}{2}} = |e_n(s)|^{\frac{p}{2}} |\sigma^i(s, X(s)) - \sigma_n^i(s, X_n(\kappa_n(s)))|^{\frac{p}{2}}$$

which by using Young's inequality becomes

$$\begin{aligned}
|\langle e_n(s), \bar{\sigma}_n^i(s) \rangle|^{\frac{p}{2}} &\leq 3^{\frac{p-2}{2}} \left\{ C_R^{\frac{p}{4}} |e_n(s)|^p + C_R^{\frac{p}{4}} |e_n(s)|^{\frac{p}{2}} |X_n(s) - X_n(\kappa_n(s))|^{\frac{p}{2}} \right. \\
&\quad \left. + |e_n(s)|^{\frac{p}{2}} |\sigma^i(s, X_n(\kappa_n(s))) - \sigma_n^i(s, X_n(\kappa_n(s)))|^{\frac{p}{2}} \right\} \\
&\leq \frac{3^{\frac{p-2}{2}}}{2} \left\{ (3C_R^{\frac{p}{4}} + 1) |e_n(s)|^p + C_R^{\frac{p}{4}} |X_n(s) - X_n(\kappa_n(s))|^p \right. \\
&\quad \left. + |\sigma^i(s, X_n(\kappa_n(s))) - \sigma_n^i(s, X_n(\kappa_n(s)))|^p \right\}.
\end{aligned}$$

Thus

$$\begin{aligned}
IV &:= 2^{\frac{3p-4}{2}} m^{\frac{p-2}{2}} T^{\frac{p-4}{4}} \bar{c} \sum_{i=1}^m \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_{nR}} |\langle e_n(s), \bar{\sigma}_n^i(s) \rangle|^{\frac{p}{2}} ds \right] \\
&\leq 2^{\frac{3(p-2)}{2}} 3^{\frac{p-2}{2}} m^{\frac{p-2}{2}} T^{\frac{p-4}{4}} \bar{c} \left\{ m \left(3f(R)^{\frac{p}{4}} + 1 \right) \int_{t_0}^u \mathbb{E} \left[\sup_{t_0 \leq r \leq s} |e_n(r \wedge \nu_{nR})|^p \mathbb{1}_{\{C_R \leq f(R)\}} \right] ds \right. \\
&\quad \left. + mf(R)^{\frac{p}{4}} \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] \right. \\
&\quad \left. + \sum_{i=1}^m \mathbb{E} \left[\int_{t_0}^{t_1} |\sigma^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - \sigma_n^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \right\} \quad (19)
\end{aligned}$$

Hence using (17), (18) and (19) in (15) and then after applying Gronwall's inequality, one obtains

$$\begin{aligned}
\mathbb{E} \left[\sup_{t_0 \leq t \leq t_1} |e_n(t \wedge \nu_{nR})|^p \mathbb{1}_{\{C_R \leq f(R)\}} \right] &\leq e^{\bar{K}_1 T} \left\{ 2^{p-2} \mathbb{E}[|e_n(t_0)|^p] \right. \\
&\quad + \bar{K}_2 \mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right] \\
&\quad + \bar{K}_3 \mathbb{E} \left[\int_{t_0}^{t_1} |b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - b_n(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \\
&\quad + \bar{K}_5 \sum_{i=1}^m \mathbb{E} \left[\int_{t_0}^{t_1} |\sigma^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - \sigma_n^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \Big\} \\
&\quad + \bar{K}_4 \sqrt{\mathbb{E} \left[\int_{t_0}^{t_1} |X_n(s \wedge \nu_{nR}) - X_n(\kappa_n(s \wedge \nu_{nR}))|^p ds \right]} \tag{20}
\end{aligned}$$

where constants $\bar{K}_1, \bar{K}_2, \bar{K}_3, \bar{K}_4$ and \bar{K}_5 are appropriately defined and depend explicitly on $f(R)$ but not on R and they are independent of n . Thus one can choose η sufficiently small and R sufficiently large such that for $\epsilon > 0$ (however small),

$$\frac{\eta p}{q} 2^q K < \frac{\epsilon}{3}, \quad \frac{q-p}{q\eta^{p/(q-p)}} \frac{2K}{R^p} < \frac{\epsilon}{3} \quad \text{and} \quad \frac{q-p}{q\eta^{p/(q-p)}} \mathbb{P}(C_R > f(R)) < \frac{\epsilon}{3}.$$

Therefore, substituting equations (14) and (20) in (13) and then using assumption A-3, A-4 and Lemma 2, one obtains (12) for $p > 4$. The application of Hölder's inequality completes the proof. \square

4 Proof of Main Result

In this section, we shall prove Theorem 1 by considering SDDE (1) as a special case of an SDE with random coefficients. In particular, we set

$$b(t, x) := \beta(t, Y(t), x), \sigma(t, x) := \alpha(t, Y(t), x), b_n(t, x) := \beta(t, Y_n(t), x), \sigma_n(t, x) := \alpha(t, Y_n(t), x) \tag{21}$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$.

Proof of theorem 1. We shall prove the result using an induction method so as to show that,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sup_{(i-1)\tau \leq t \leq i\tau} |X(t) - X_n(t)|^p \right] = 0 \tag{22}$$

for every $i \in \{1, \dots, N\}$ and for all $p > 0$.

Case: $\mathbf{t} \in [0, \tau]$. When $t \in [0, \tau]$, then SDDE (1) and Euler scheme (2) become SDE (4) and scheme (5) with $t_0 = 0$, $t_1 = \tau$, $X(0) = X_n(0) = \xi(0)$ and coefficients given by (21). Also one observes that A-1 to A-5 hold due to C-1 to C-3. In particular, A-1 is a consequence of C-1 with $M = M_n = 1 + \Psi^l$, where $\Psi := \sup_{t \in [0, \tau]} |(\xi(\delta_1(t)), \dots, \xi(\delta_k(t)))|$, which is uniformly bounded due to the fact that $\xi \in C_{\mathcal{F}_0}^b$. Further, A-2 is a consequence of C-2 since one observes that C_R is deterministic and in fact $C_R = f(R)$, where $f(R) = K_\Psi \mathbb{1}_{\{0 < R < \Psi\}} + K_R \mathbb{1}_{\{R \geq \Psi\}}$ and therefore $\mathbb{P}(C_R > f(R)) = 0$ for any $R > 0$. Finally, A-3, A-4 and A-5 hold trivially. Therefore, for $i = 1$, equation (22) holds due to Theorem 2 and Lemma 1 for $t_0 = 0$ and $t_1 = \tau$.

For inductive arguments, we assume that when $i = r$, i.e. $t \in [(r-1)\tau, r\tau]$ for some $r \in \{1, \dots, N-1\}$, equation (22) is satisfied and Lemma 1 holds for $t_0 = (r-1)\tau$ and $t_1 = r\tau$. Then we claim that when $i = r+1$, i.e. $t \in [r\tau, (r+1)\tau]$, equation (22) holds and Lemma 1 is true for $t_0 = r\tau$ and $t_1 = (r+1)\tau$.

Case: $\mathbf{t} \in [r\tau, (r+1)\tau]$. When $t \in [r\tau, (r+1)\tau]$, then SDDE (1) and its Euler scheme (2) become SDE (4) and scheme (5) with $t_0 = r\tau$, $t_1 = (r+1)\tau$, $X(t_0) = X(r\tau)$, $X_n(t_0) = X_n(r\tau)$ and coefficients given by (21).

Verify A-1. Consider for $t \in [r\tau, (r+1)\tau]$ and $x \in \mathbb{R}^d$, then due to C-1,

$$|b(t, x)| + |\sigma(t, x)| = |\beta(t, Y(t), x)| + |\alpha(t, Y(t), x)| \leq G(M + |x|)$$

and

$$|b_n(t, x)| + |\sigma_n(t, x)| = |\beta(t, Y_n(t), x)| + |\alpha(t, Y_n(t), x)| \leq G(M_n + |x|)$$

where $M := 1 + \sup_{r\tau \leq t \leq (r+1)\tau} |Y(t)|^l$ and $M_n := 1 + \sup_{r\tau \leq t \leq (r+1)\tau} |Y_n(t)|^l$ which are bounded in \mathcal{L}^p for every $p > 0$ because of Lemma 1 and the inductive assumptions.

Verify A-2. For every $R > 0$, $|x|, |z| \leq R$ and $t \in [r\tau, (r+1)\tau]$, from C-2, one obtains,

$$\begin{aligned} \langle x - z, b(t, x) - b(t, z) \rangle &= \langle x - z, \beta(t, Y(t), x) - \beta(t, Y(t), z) \rangle \leq C_R |x - z|^2 \\ |\sigma(t, x) - \sigma(t, z)|^2 &= |\alpha(t, Y(t), x) - \alpha(t, Y(t), z)|^2 \leq C_R |x - z|^2 \end{aligned}$$

where the random variable C_R is given by

$$C_R := K_R 1_{\Omega_R} + \sum_{j=R}^{\infty} K_{j+1} 1_{\{\Omega_{j+1} \setminus \Omega_j\}}$$

with $\Omega_j := \{\omega \in \Omega : \sup_{t \in [r\tau, (r+1)\tau]} |Y(t)| \leq j\}$. One observes that C_R is an $\mathcal{F}_{r\tau}$ -measurable random variable. Further one takes $f(R) := K_R$ for every $R > 0$ so that

$$\begin{aligned} \mathbb{P}(C_R > f(R)) &= \mathbb{P}(C_R > K_R) = 1 - \mathbb{P}(C_R \leq K_R) \\ &\leq 1 - \mathbb{P}(\Omega_R) = \mathbb{P}\left(\sup_{r\tau \leq t < (r+1)\tau} |Y(t)| > R\right) \rightarrow 0 \text{ as } R \rightarrow \infty. \end{aligned} \quad (23)$$

Verify A-3. For $R > 0$ and $|x| \leq R$, one observes that $|Y(t) - Y_n(t)| \xrightarrow{\mathbb{P}} 0$ as $\mathbb{E}[|Y(t) - Y_n(t)|^p] \rightarrow 0$ for every $p > 0$ when $n \rightarrow \infty$. Therefore from C-3,

$$\sup_{|x| \leq R} |\beta(t, Y_n(t), x) - \beta(t, Y(t), x)|^p \xrightarrow{\mathbb{P}} 0 \quad \text{and} \quad \sup_{|x| \leq R} |\alpha(t, Y_n(t), x) - \alpha(t, Y(t), x)|^p \xrightarrow{\mathbb{P}} 0.$$

Furthermore, one observes that sequences

$$\left\{ \sup_{|x| \leq R} |\beta(t, Y_n(t), x) - \beta(t, Y(t), x)|^p \right\}_{n \geq 1} \quad \text{and} \quad \left\{ \sup_{|x| \leq R} |\alpha(t, Y_n(t), x) - \alpha(t, Y(t), x)|^p \right\}_{n \geq 1}$$

are uniformly integrable since they are bounded in \mathcal{L}^q for any $q > 1$,

$$\begin{aligned} \mathbb{E} \left[\sup_{|x| \leq R} |\beta(t, Y_n(t), x) - \beta(t, Y(t), x)|^{pq} \right] &= \mathbb{E} \left[\sup_{|x| \leq R} |\beta(t, Y_n(t), x) - \beta(t, Y(t), x)|^{pq} \right] \\ &\leq 2^{pq-1} \mathbb{E} \left[\sup_{|x| \leq R} |\beta(t, Y_n(t), x)|^{pq} \right] + 2^{pq-1} \mathbb{E} \left[\sup_{|x| \leq R} |\beta(t, Y(t), x)|^{pq} \right] \\ &\leq 6^{pq-1} G^{pq} \{1 + \mathbb{E}[|Y_n(t)|^{lpq}] + |R|^{pq}\} + 6^{pq-1} G^{pq} \{1 + \mathbb{E}[|Y(t)|^{lpq}] + |R|^{pq}\} \\ &\leq 6^{pq-1} G^{pq} 2 \{1 + K + |R|^{pq}\} \end{aligned}$$

and similarly, for the sequence $\{\sup_{|x| \leq R} |\alpha(t, Y_n(t), x) - \alpha(t, Y(t), x)|^p\}_{n \geq 1}$. Therefore,

$$\mathbb{E} \left[\sup_{|x| \leq R} \left\{ |\beta(t, Y_n(t), x) - \beta(t, Y(t), x)|^p + |\alpha(t, Y_n(t), x) - \alpha(t, Y(t), x)|^p \right\} \right] \rightarrow 0$$

as $n \rightarrow \infty$ due to Dominated Convergence Theorem which also yields

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_{r\tau}^{(r+1)\tau} \sup_{|x| \leq R} \{ |b(s, x) - b_n(s, x)|^p + |\sigma(s, x) - \sigma_n(s, x)|^p \} ds \right] = 0.$$

Thus A-3 is satisfied.

Verify A-4. This holds due to the inductive assumptions.

Verify A-5. This follows due to the inductive assumptions.

Finally Lemma 1 holds for $t_0 = r\tau$ and $t_1 = (r+1)\tau$ due to the fact that A-1 holds for $t \in [r\tau, (r+1)\tau]$ and $x \in \mathbb{R}^d$ and A-5 is true for $t_0 = r\tau$. This completes the proof. \square

5 Rate of Convergence

In this section, we shall recover the rates of convergence of the Euler scheme (2) under different set of assumptions. First we state below the relevant assumptions.

C-4. There exist constants $C > 0$ and $l_1 > 0$ such that for any $t \in [0, T]$,

$$\begin{aligned} \langle x - x', \beta(t, y, x) - \beta(t, y, x') \rangle \vee |\alpha(t, y, x) - \alpha(t, y, x')|^2 &\leq C|x - x'|^2 \\ |\beta(t, y, x) - \beta(t, y', x)|^2 + |\alpha(t, y, x) - \alpha(t, y', x)|^2 &\leq C(1 + |y|^{l_1} + |y'|^{l_1})|y - y'|^2 \end{aligned}$$

for all $x, x' \in \mathbb{R}^d$ and $y, y' \in \mathbb{R}^{d \times k}$.

C-5. There exist constants $C > 0$ and $l_2 > 0$ such that for any $t \in [0, T]$,

$$|\beta(t, y, x) - \beta(t, y, x')|^2 \leq C(1 + |x|^{l_2} + |x'|^{l_2})|x - x'|^2$$

for all $x, x' \in \mathbb{R}^d$ and $y \in \mathbb{R}^{d \times k}$.

In Corollary 1, we prove that under C-1 and C-4, the rate of convergence of Euler scheme (2) is one-fourth. On the other hand, in Corollary 2 we prove that if one makes assumption C-5 in addition to C-1 and C-4, then the classical rate of convergence (one-half) can be recovered (see also [11], [14], [15], [16] and [20]).

Corollary 1. *Suppose C-1 and C-4 hold, then for all $p > 0$, the Euler scheme (2) converges to the true solution (1) in \mathcal{L}^p -sense with rate $1/4$, i.e.*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \leq \bar{C} n^{-\frac{p}{4}}$$

where the constant $\bar{C} > 0$ does not depend on n .

Proof. In equation (14), we choose $\eta = n^{-\frac{p}{2}}$, $R = n^{\frac{q-2p}{2(q-p)}}$, $q > 2p \geq 4$, then

$$\frac{\eta p}{q} 2^q K = \frac{pK2^q}{q} n^{-\frac{p}{2}}, \quad \frac{q-p}{q\eta^{p/(q-p)}} \frac{2K}{R^p} = \frac{2(q-p)K}{q} n^{-\frac{p}{2}} \text{ and } \frac{q-p}{q\eta^{p/(q-p)}} \mathbb{P}(C_R > f(R)) \equiv 0 \quad (24)$$

where we choose $C_R \equiv C$ and $f(R) \equiv C$. Then one can show that,

$$\mathbb{E} \left[\sup_{(k-1)\tau \leq t \leq k\tau} |X(t) - X_n(t)|^p \right] \leq \bar{C} n^{-\frac{p}{4}} \quad (25)$$

for each $k \in \{1, \dots, N\}$. As seen before, SDDE (1) corresponds to the ordinary SDE (4) with $X(0) = X_n(0) = \xi(0)$, $Y(t) = Y_n(t) = \xi(\delta(t))$ for all $t \in [0, \tau]$. Therefore, one obtains the rate as in (25) for $k = 1$, which is $1/4$ instead of $1/2$, due to the last term of (17), that is also the last term in (20), the estimate in (11) and the fact that

$$\mathbb{E} \left[\int_0^\tau |b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - b_n(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \equiv 0$$

and

$$\mathbb{E} \left[\int_0^\tau |\sigma^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - \sigma_n^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \equiv 0$$

for every $i \in \{1, \dots, m\}$.

One then assumes that (25) holds for $k = r$, i.e. $t \in [(r-1)\tau, r\tau]$ for some $r \in \{1, \dots, N-1\}$, so as to show that it also holds for $t \in [r\tau, (r+1)\tau]$. Due to inductive assumption, the initial data of SDE (4) satisfy

$$\mathbb{E} [|X(r\tau) - X_n(r\tau)|^p] \leq \bar{C} n^{-\frac{p}{4}} \quad (26)$$

and due to (21), C-4, Lemma 1 and inductive assumptions, one obtains the following estimates,

$$\begin{aligned}
& \mathbb{E} \left[\int_{r\tau}^{(r+1)\tau} |b(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - b_n(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \\
&= \mathbb{E} \left[\int_{r\tau}^{(r+1)\tau} |\beta(s, Y(s), X_n(\kappa_n(s))) - \beta(s, Y_n(s), X_n(\kappa_n(s)))|^p \mathbb{1}_{\{s \leq \nu_{nR}\}} ds \right] \\
&\leq \int_{r\tau}^{(r+1)\tau} \mathbb{E} [(|1 + |Y(s)|^{l_1} + |Y_n(s)|^{l_2})^{\frac{p}{2}} |Y(s) - Y_n(s)|^p] ds \\
&\leq 3^{\frac{p-1}{2}} \sqrt{(1+2K)} \int_{r\tau}^{(r+1)\tau} \sqrt{\mathbb{E}[|Y(s) - Y_n(s)|^{2p}]} ds \\
&\leq \bar{C} n^{-\frac{p}{2}}.
\end{aligned} \tag{27}$$

One similarly estimates

$$\mathbb{E} \left[\int_{r\tau}^{(r+1)\tau} |\sigma^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR}))) - \sigma_n^i(s \wedge \nu_{nR}, X_n(\kappa_n(s \wedge \nu_{nR})))|^p ds \right] \leq \bar{C} n^{-\frac{p}{2}} \tag{28}$$

for every $i \in \{1, \dots, m\}$. Therefore, (25) holds due to (20) by taking into account (11), (24), (26), (27) and (28) for $t \in [r\tau, (r+1)\tau]$ and $p \geq 4$. Then the application of Hölder's inequality completes the proof. \square

Corollary 2. *Suppose C-1, C-4 and C-5 hold, then for all $p > 0$, the Euler scheme (2) converges to the true solution (1) in \mathcal{L}^p -sense with rate $1/2$, i.e.*

$$\mathbb{E} \left[\sup_{0 \leq t \leq T} |X(t) - X_n(t)|^p \right] \leq \bar{C} n^{-\frac{p}{2}}$$

where the constant $\bar{C} > 0$ does not depend on n .

Proof. In order to prove this, one observes that the estimates of the rate of convergence of terms of II in (17) is improved if C-5 is used along with C-4. For this, one could replace the right hand side of (16) by the following,

$$\begin{aligned}
\langle e_n(s), \bar{b}_n(s) \rangle &= \langle e_n(s), b(s, X(s)) - b(s, X_n(s)) \rangle + \langle e_n(s), b(s, X_n(s)) - b(s, X_n(\kappa_n(s))) \rangle \\
&\quad + \langle e_n(s), b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s))) \rangle
\end{aligned}$$

which on using (21) becomes

$$\begin{aligned}\langle e_n(s), \bar{b}_n(s) \rangle &= \langle e_n(s), \beta(s, Y(s), X(s)) - \beta(s, Y(s), X_n(s)) \rangle \\ &\quad + \langle e_n(s), \beta(s, Y(s), X_n(s)) - \beta(s, Y(s), X_n(\kappa_n(s))) \rangle \\ &\quad + \langle e_n(s), \beta(s, Y(s), X_n(\kappa_n(s))) - \beta(s, Y_n(s), X_n(\kappa_n(s))) \rangle.\end{aligned}$$

By using C-4 and Cauchy-Schwartz and Young's inequalities, one obtains

$$\begin{aligned}\langle e_n(s), \bar{b}_n(s) \rangle &\leq (C+1)|e_n(s)|^2 + \frac{1}{2}|\beta(s, Y(s), X_n(s)) - \beta(s, Y(s), X_n(\kappa_n(s)))|^2 \\ &\quad + \frac{1}{2}|\beta(s, Y(s), X_n(\kappa_n(s))) - \beta(s, Y_n(s), X_n(\kappa_n(s)))|^2\end{aligned}$$

which due to C-4 and C-5 yields

$$\begin{aligned}\langle e_n(s), \bar{b}_n(s) \rangle &\leq (C+1)|e_n(s)|^2 + \frac{C}{2}\{1 + |X_n(s)|^{l_2} + |X_n(\kappa_n(s))|^{l_2}\}|X_n(s) - X_n(\kappa_n(s))|^2 \\ &\quad + \frac{C}{2}\{1 + |Y(s)|^{l_1} + |Y_n(s)|^{l_1}\}|Y(s) - Y_n(s)|^2.\end{aligned}$$

Therefore, one estimates II in (17) by the following,

$$\begin{aligned}II &:= (32T)^{\frac{p-2}{2}} \mathbb{E} \left[\mathbb{1}_{\{C_R \leq f(R)\}} \int_{t_0}^{u \wedge \nu_n R} |\langle e_n(s), \bar{b}_n(s) \rangle|^{\frac{p}{2}} ds \right] \\ &\leq (96T)^{\frac{p-2}{2}} \left\{ (C+1)^{\frac{p-2}{2}} \int_{r\tau}^u \mathbb{E} \left[\sup_{r\tau \leq t \leq s} |e_n(s)|^p \right] ds \right. \\ &\quad + \left(\frac{C}{2} \right)^{\frac{p}{2}} \mathbb{E} \left[\int_{r\tau}^u \{1 + |X_n(s)|^{l_2} + |X_n(\kappa_n(s))|^{l_2}\}^{\frac{p}{2}} |X_n(s) - X_n(\kappa_n(s))|^p ds \right] \\ &\quad \left. + \left(\frac{C}{2} \right)^{\frac{p}{2}} \mathbb{E} \left[\int_{r\tau}^u \{1 + |Y(s)|^{l_1} + |Y_n(s)|^{l_1}\}^{\frac{p}{2}} |Y(s) - Y_n(s)|^p ds \right] \right\}\end{aligned}$$

which on the application of Hölder's inequality and Lemma 1 yields

$$\begin{aligned}II &\leq (96T)^{\frac{p-2}{2}} \left\{ (C+1)^{\frac{p-2}{2}} \int_{r\tau}^u \mathbb{E} \left[\sup_{r\tau \leq t \leq s} |e_n(s)|^p \right] ds \right. \\ &\quad \left. + \left(\frac{C}{2} \right)^{\frac{p}{2}} \hat{K} \int_{r\tau}^{(r+1)\tau} \left[\sqrt{\mathbb{E} [|X_n(s) - X_n(\kappa_n(s))|^{2p}]} + \sqrt{\mathbb{E} [|Y(s) - Y_n(s)|^{2p}]} \right] ds \right\}. \quad (29)\end{aligned}$$

where \hat{K} is a constant which does not depend on n . Therefore, in (15), one replaces the estimate (17) of II by the estimate (29) whereas III and IV remain the same. Then the first term of (29) is incorporated with similar terms of III and IV so as to apply Gronwall's inequality. Furthermore, one observes that the second and third terms of (29) are of (improved) order $\mathcal{O}(n^{-p/2})$ due to Lemma 2 and inductive assumptions respectively. Thus, by adopting same arguments as adopted in the proof of Corollary 1 with $\eta = n^{-\frac{p}{2}}$, one obtains the desired rate (1/2). \square

6 Numerical Examples

In this section, we shall illustrate our findings with the help of numerical examples. We consider the following SDDE given by

$$dZ(t) = [aZ(t) + b\{Z(t-\tau)\}^{l_1}]dt + [\beta_1 + \beta_2 Z(t) + \beta_3\{Z(t-\tau)\}^{l_2}]dW(t) \quad \text{for } t \in [0, 2\tau] \quad (30)$$

with initial data $\xi(t)$ when $t \in [-\tau, 0]$, $\tau > 0$ is a fixed delay, $l_1, l_2 > 0$, $a, b, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$.

When $t \in [0, \tau]$, then explicit solution of SDE (30) is given by,

$$Z(t) = \Phi_{0,t} \left\{ \xi(0) + \int_0^t \Phi_{0,s}^{-1} \{ b\{\xi(s-\tau)\}^{l_1} - \beta_2(\beta_1 + \beta_3\{\xi(s-\tau)\}^{l_2}) \} ds + \int_0^t \Phi_{0,s}^{-1} \{ \beta_1 + \beta_3\{\xi(s-\tau)\}^{l_2} \} dW(s) \right\}, \quad (31)$$

where,

$$\Phi_{0,t} = \exp \left(\left\{ a - \frac{\beta_2^2}{2} \right\} t + \beta_2 W(t) \right).$$

And when $t \in [\tau, 2\tau]$, then the explicit solution is given by

$$Z(t) = \Phi_{\tau,t} \left\{ Z(\tau) + \int_\tau^t \Phi_{\tau,s}^{-1} [b\{Z(s-\tau)\}^{l_1} - \beta_2(\beta_1 + \beta_3\{Z(s-\tau)\}^{l_2})] ds + \int_\tau^t \Phi_{\tau,s}^{-1} (\beta_1 + \beta_3\{Z(s-\tau)\}^{l_2}) dW(s) \right\} \quad (32)$$

where,

$$\Phi_{\tau,t} = \exp \left(\left\{ a - \frac{\beta_2^2}{2} \right\} (t - \tau) + \beta_2 (W(t) - W(\tau)) \right).$$

For Euler scheme, we divide the interval $[0, \tau]$ into sub-intervals of width $\frac{\tau}{2^N}$ for some positive integer N . Then solution at kh -th grid is given by

$$\begin{aligned} Z_n((k+1)h) &= [aZ_n(kh) + b\{\xi(kh - \tau)\}^{l_1}]h \\ &\quad + \left[\beta_1 + \beta_2 Z_n(kh) + \beta_3 \{\xi(kh - \tau)\}^{l_2} \right] (W((k+1)h) - W(kh)) \end{aligned}$$

with $\xi(t)$ for $t \in [-\tau, 0]$, $k = 1, \dots, 2^N$. We use this solution to find the solution of the Euler scheme in the interval $[\tau, 2\tau]$.

We take values of the parameters of SDDE (30) given in Table 1. From Table 2, we observe that as step size is decreased, the error is decreased too. In Figure 1, the reference line has a slope of -0.5 .

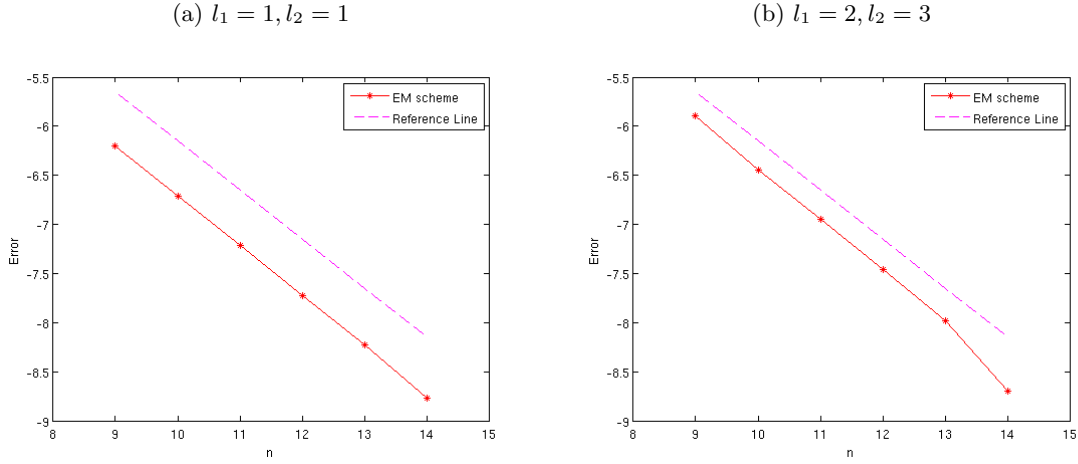
Table 1: Values of Parameters

p	τ	a	b	β_1	β_2	β_3	$\xi(t)$ for $t \in [-\tau, 0]$
2	1	-8	4	0	1	1	t+1

Table 2: Effect of decreasing step size on Error

h	$\sqrt{\mathbb{E}(Z(T) - Z_n(T) ^2)}$		
	$l_1 = 1/2, l_2 = 1/2$	$l_1 = 1, l_2 = 1$	$l_1 = 2, l_2 = 3$
2^{-9}	0.0002332678715832590	0.0001857299504105190	0.0002812168347124360
2^{-10}	0.0001171164266162860	0.0000911946799085898	0.0001308779599825470
2^{-11}	0.0000605972526234064	0.0000450951550271487	0.0000652194083354262
2^{-12}	0.0000314672425957171	0.0000223376219662284	0.0000321943295689175
2^{-13}	0.0000159827178333670	0.0000111954499489959	0.0000156672183324178
2^{-14}	0.0000079146081811971	0.0000052537027007630	0.0000057778221373887
slope		-0.51157	-0.54615

Figure 1: Strong order of convergence of the Euler scheme.



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